Non-truncated formal series symmetries and generalized Virasoro algebra: AKNS-type breaking soliton equation

1995 J. Phys. A: Math. Gen. 28 5943

(http://iopscience.iop.org/0305-4470/28/20/022)

J. Phys. A: Math. Gen. 28 (1995) 5943–5949. Printed in the UK

Non-truncated formal series symmetries and generalized Virasoro algebra: AKNS-type breaking soliton equation

Sen-yue Lou

CCAST (World Laboratory), PO Box 8730, Beijing 100080, People's Republic of China
Institute of Modern Physics, Ningbo Normal College, Ningbo, 315211, People's Republic of China†
Department of Physics, Fudan University, Shanghai 200433, People's Republic of China

Received 30 January 1995, in final form 10 August 1995

Abstract. Using the formal series symmetry approach, we obtain three sets of generalized symmetries of the AKNS-type breaking soliton equation. These sets of formal series symmetries are not truncated except that the arbitrary functions of the formal series symmetries are fixed as polynomials of the corresponding independent variables. Differing from the truncated cases such as KP and Toda field equations, these non-truncated symmetries constitute another type of generalization of the Virasoro algebra.

1. Introduction

The importance of the Virasoro algebra in string theory, integrable models, 2D gravity and conformal field theory is well known. Recently, the Virasoro algebra has been extended to a more general form, W_∞ algebra, which has been applied in various modern physics fields like the $sl(\infty)$ Toda field theory [1], integrable models [2], membrane theory [3] and W string and W gravity theories [4]. In the study of the symmetries of the higher-dimensional integrable models, we have established a formal series symmetry approach to obtain generalized symmetries for a general type of higher-dimensional model. A simple formal series symmetry formula is given. For some integrable models such as the Toda field equation [5], the KP equation [6], the two-dimensional dispersive long-wave equation [7] and the Nizhnik–Novikov–Veselov equation [8], the formal series symmetries become truncated. These sets of truncated symmetries constitute some extensions of the usual W_∞ algebra [5–8] which is one type of generalization of the Virasoro algebra. However, for some other types of $(2+1)$-dimensional integrable models such as the Davey–Stewartson equation [9], the KdV–Ito equation [10] and Sawada–Kortera equation [11], only finite truncated symmetries are found and we have not found a generalized W_∞ symmetry algebra for these types of integrable models. Now the significant questions arise: whether the non-truncated symmetries are meaningful and whether the non-truncated symmetries constitute a closed symmetry algebra?

In section 2 of this paper, we would like to study the symmetries of the $(2+1)$-dimensional AKNS-type of breaking soliton equation. Three sets of formal series symmetries are found. Generally, these sets of formal series symmetries are not truncated for arbitrary functions while for some special selection of the arbitrary functions, say, polynomials of

† Mailing address.

their arguments, the series will be truncated. The full symmetry algebra of these sets of non-truncated formal series symmetries is given in section 3. Section 4 contains a summary and discussion.

2. Symmetries of the AKNS-type breaking soliton equation

In this section, we will consider the symmetry of the following $(2+1)$-dimensional AKNS-type nonlinear evolution equation

$$q_t = \mathrm{i}q_{xy} - 2\mathrm{i}q\partial_x^{-1}(qr)_y \tag{1}$$

$$r_t = -\mathrm{i}r_{xy} + 2\mathrm{i}r\partial_x^{-1}(qr)_y. \tag{2}$$

Equations (1) and (2) can be considered as the compatibility condition for the following Lax pair:

$$\psi_x = M\psi \tag{3}$$

$$\psi_t = 2\xi\psi_y + N\psi \tag{4}$$

with

$$M = \begin{pmatrix} -\mathrm{i}\xi & q \\ r & \xi \end{pmatrix}$$

$$N = \begin{pmatrix} -\mathrm{i}\partial_x^{-1}(qr)_y & \mathrm{i}q_y \\ -\mathrm{i}r_y & \mathrm{i}\partial_x^{-1}(qr)_y \end{pmatrix} \tag{5}$$

and

$$\xi_x = 0 \qquad \xi_t = 2\xi\xi_y \tag{6}$$

i.e. equations (1) and (2) have the non-isospectral zero curvature equation

$$M_t - N_x + MN - NM - 2\xi M_y = 0. \tag{7}$$

When $q = -r^* = \psi$, the equation system (1) and (2) reduces to a $(2+1)$-dimensional nonlinear Schrödinger equation

$$\mathrm{i}\psi_t + \psi_{xx} + 2\psi(\partial_x^{-1}|\psi|^2)_y = 0 \tag{8}$$

and when $y = x$, equation (8) is just the well known $(1+1)$-dimensional nonlinear Schrödinger equation. Equation (8) may be found in [12]. In [13] and [14] equation (8) was deduced from twistor space and its soliton solutions are found by the Hirota method. More solutions of (3) were given in [15] by Darboux transformation.

It is known that [16] the equation system (1) and (2) possesses a recursion operator

$$\Phi = \begin{pmatrix} -\partial_x + 2q\partial_x^{-1}r & 2q\partial_x^{-1}q \\ -2r\partial_x^{-1}r & \partial_x - 2r\partial_x^{-1}q \end{pmatrix} \tag{9}$$

which is the same as that of the $(1+1)$-dimensional AKNS system. Then using the recursion operator and the seed symmetries, say space x and y translation invariance and scaling invariance, one can get some sets of infinitely many symmetries [16] ($u \equiv \binom{q}{r}$).

$$K_n^1 = \Phi^n K_0^1 \qquad \left(K_0^1 = \begin{pmatrix} -q \\ r \end{pmatrix}\right) \qquad K_n^2 = \Phi^n u_y$$

$$\tau_n^2 = \Phi^n(tu_t + yu_y) \qquad \tau_n^1 = \Phi^{n-1}(-xu_x - u + yu_y). \tag{10}$$

Using the 'dressing method' [17], the authors of [18] have obtained all t^k ($k > 0$) dependent symmetries. In this paper we manage to derive more generalized symmetries of the equation system (1) and (2) by means of a much simpler method given in [5–8]. For simplicity in the notation, we rewrite equations (1) and (2) as

$$u_t = \Phi u_y \equiv K(q, r) \equiv K(u). \tag{11}$$

A symmetry of equation (11), σ, is defined as the solution of the linearized equation (11).

$$\sigma_t \equiv \begin{pmatrix} \delta q \\ \delta r \end{pmatrix}_t = K'\sigma \equiv \lim_{\epsilon \to 0} \frac{\partial}{\partial \epsilon} K(q + \epsilon\delta q,\ r + \epsilon \delta r) \tag{12}$$

which means equation (11) is invariant in form under the transformation $u \to u + \epsilon\sigma$ with an infinitesimal parameter ϵ. Now we look for the solutions of equation (12) having the form

$$\sigma = \sum_{k=0}^{\infty} f^{(k)}(y)\sigma[k] \tag{13}$$

where $f(y)$ is an arbitrary function of y, $f^{(k)}(y) = \frac{d^k}{dy^k} f(y)$ and $\sigma[k]$ should be determined later. Substituting equation (13) into equation (12), we have

$$\sum_{k=0}^{\infty} f^{(k)}(y)\sigma_t[k] = \sum_{k=o}^{\infty} f^{(k)}(y) K'\sigma[k] + \sum_{k=1}^{\infty} f^{(k)}(y)\Phi\sigma[k-1]. \tag{14}$$

Because f is an arbitrary function of y, equation (14) should be true at any order of derivatives of $f(y)$. That means

$$\sigma_t[0] = K'\sigma[0] \tag{15}$$

$$\sigma_t[k] - K'\sigma[k] = \Phi\sigma[k-1] \qquad (k \geqslant 0). \tag{16}$$

Equation (15) shows us that $\sigma[0]$ itself should be a symmetry of equation (14), while $\sigma[k]$ $(k > 0)$ should be determined recursively from equation (16). Fortunately, because $\sigma[0]$ is a symmetry and Φ is a recursion operator of the model, equation (16) can be solved easily. The result reads:

$$\sigma[k] = \frac{t^k}{k!}\Phi^k \sigma[0]. \tag{17}$$

Substituting equation (17) into (13) we can get a formal series symmetry starting from every one of known symmetry:

$$\sigma(f) = \sum_{k=0}^{\infty} f^{(k)}(y) \frac{t^k}{k!} \Phi^k \sigma[0]. \tag{18}$$

Generally speaking, series (18) is not truncated for the general function $f(y)$. However, if $f(y)$ is fixed as a polynomial of y, say, $f = y^n$, then $\sigma(f)$ becomes truncated. For instance:

$$\sigma(y^n) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} t^k y^{n-k} \Phi^k \sigma[0]. \tag{19}$$

Using the known symmetries given in [16] or [18], we can get various sets of formal series symmetries. However, many of them are not independent. The only three sets of independent formal series symmetries read:

$$K_n^1(f) = \sum_{k=0}^{\infty} f^{(k)} \frac{t^k}{k!} K_{n+k}^1 = \sum_{k=0}^{\infty} f^{(k)} \frac{t^k}{k!} \Phi^{n+k-1} u_x \tag{20}$$

$$K_m^2(f) = \sum_{k=0}^{\infty} f^{(k)} \frac{t^k}{k!} K_{m+k}^2 = \sum_{k=0}^{\infty} f^{(k)} \frac{t^k}{k!} \Phi^{m+k} u_y \tag{21}$$

$$\tau_m(f) = \sum_{k=0}^{\infty} f^{(k)} \frac{t^k}{k!} \tau^1_{m+k} = \sum_{k=0}^{\infty} f^{(k)} \frac{t^k}{k!} \Phi^{m+k-1}(-(xu)_x + yu_y) \tag{22}$$

where $n = 0, 1, 2, \ldots$ because of $K^1_{-m} = 0$ ($m \leqslant 0$) and $m = 0, \pm 1, \pm 2, \ldots$ because of Φ being invertible [19] and $K^2_{-m} \neq 0$, $\tau^1_{-m} \neq 0$. If we restrict f as polynomials of y, then all the symmetries of [16] and [18] can be obtained as linear combinations of (20)–(22) with $f = y^r$ ($r = 0, 1, 2, \ldots$). For instance, $\tau^2_m = K^1_m(y)$, $\tau^1_m = \tau_m(1)$, $K^1_n = K^1_n(1)$ and $K^2_m = K^2_m(1)$. At first sight, one may use (20)–(22) as new seed symmetries to get new symmetries by substituting them into equation (18), but this procedure does not yield any new result because we can redefine the arbitrary function of (18). We omit the proof of this conclusion due to its simplicity.

3. Algebra structure of the non-truncated formal series symmetries of the AKNS-type breaking soliton equation

In this section, we shall consider the algebraic structure of the symmetries of the equation system (1) and (2) obtained in the last section. The Lie product of two symmetries A and B is defined as

$$[A, B] = A'B - B'A \equiv \lim_{\epsilon \to 0} \frac{\partial}{\partial \epsilon}[A(u + \epsilon B) - B(u + \epsilon A)]. \tag{23}$$

In order to find the algebraic structures of the formal series symmetries $K^1_n(f)$, $K^2_n(f)$ and $\tau_n(f)$, we write down the known algebraic structure of symmetries K^1_n, K^2_n and τ^1_n which have been given by Li [16]

$$[K^1_n, K^1_m] = [K^1_n, K^2_m] = [K^2_n, K^2_m] = 0 \tag{24}$$

$$[\tau^1_n, K^1_m] = mK^1_{m+n-1} \tag{25}$$

$$[\tau^1_n, K^2_m] = (m-1)K^2_{m+n-1} \tag{26}$$

$$[\tau^1_n, \tau^1_m] = (m-n)\tau^1_{m+n-1}. \tag{27}$$

Using equations (24)–(27), we can get the full symmetry algebra of $K^1_n(f)$, $K^2_n(f)$ and $\tau_n(f)$:

$$[K^1_n(f_1), K^1_m(f)] = 0 \tag{28}$$

$$[K^1_n(f), K^2_m(g)] = -K^1_{m+n}(\dot{f}g) \tag{29}$$

$$[K^2_n(f_1), K^2_m(f_2)] = K^2_{m+n}(f_1\dot{f}_2 - f_2\dot{f}_1) \tag{30}$$

$$[\tau_n(f), K^1_m(g)] = mK^1_{m+n-1}(fg) + K^1_{m+n-1}(yf\dot{g}) \tag{31}$$

$$[\tau_n(f), K^2_m(g)] = (m-1)K^2_{m+n-1}(fg) + K^2_{m+n-1}(yf\dot{g}) - \tau_{m+n}(\dot{f}g) \tag{32}$$

$$[\tau_n(f), \tau_m(g)] = (m-n)\tau_{m+n-1}(fg) + \tau_{m+n-1}(\dot{g}fy - \dot{f}gy) \tag{33}$$

where $\dot{f} = \frac{\mathrm{d}}{\mathrm{d}y} f(y)$.

The derivation of the commutation relations (28)–(33) is rather straightforward. Here we only give the derivation of the commutation relation (30), while the others can be derived in a similar way.

$$\begin{aligned}
[K^2_n(f_1), K^2_m(f_2)] &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left[\frac{f_1^{(k)}t^{k+l}}{k!l!}(K^2_{n+k})'(f_2^{(l)}K^2_{m+l}) - \frac{f_2^{(l)}t^{k+l}}{k!l!}(K^2_{m+l})'(f_1^{(k)}K^2_{n+k})\right] \\
&= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left\{\frac{f_1^{(k)}t^{k+l}}{k!l!}\left[(\Phi^{n+k})'[f_2^{(l)}K^2_{m+l}]u_y + \Phi^{n+k}\partial_y(f_2^{(l)}K^2_{m+l})\right]\right.
\end{aligned}$$

$$
\begin{aligned}
&\left.-\frac{f_2^{(l)}t^{k+l}}{k!l!}\left[(\Phi^{m+l})'[f_1^{(k)}K_{n+k}^2]u_y+\Phi^{m+l}\partial_y(f_1^{(k)}K_{n+k}^2)\right]\right\} \\
&=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{f_1^{(k)}f_2^{(l)}t^{k+l}}{k!l!}\left[(\Phi^{n+k})'[K_{m+l}^2]u_y+\Phi^{n+k}\partial_y K_{m+l}^2\right. \\
&\quad\left.-(\Phi^{m+l})'[K_{n+k}^2]u_y-\Phi^{m+l}\partial_y K_{n+k}^2\right] \\
&\quad+\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left(\frac{f_1^{(k)}f_2^{(l+1)}t^{k+l}}{k!l!}\Phi^{n+k}K_{m+l}^2-\frac{f_1^{(k+1)}f_2^{(l)}t^{k+l}}{k!l!}\Phi^{m+l}K_{n+k}^2\right) \\
&=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{f_1^{(k)}f_2^{(l)}t^{k+l}}{k!l!}[K_{n+k}^2,\ K_{m+l}^2] \\
&\quad+\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{t^{k+l}}{k!l!}(f_1^{(k)}f_2^{(l+1)}-f_2^{(l)}f_1^{(k+1)})K_{n+m+l+k}^2 \\
&\overset{(24)}{=}\sum_{k=0}^{\infty}\frac{t^k}{k!}\left(\sum_{l=0}^{k}\frac{k!}{l!(k-l)!}(f_1^{(k-l)}f_2^{(l+1)}-f_2^{(k-l)}f_1^{(l+1)})\right)K_{m+n+k}^2 \\
&=\sum_{k=0}^{\infty}\frac{t^k}{k!}(\dot f_2 f_1-\dot f_1 f_2)^{(k)}K_{m+n+k}^2 \\
&=K_{m+n}^2(\dot f_2 f_1-\dot f_1 f_2).
\end{aligned}
\tag{34}
$$

Now, we see that, though the formal series symmetries are not truncated for general function $f(y)$, we can still get a closed symmetry algebra. In particular two types of generalized Virasoro algebra exist for non-truncated series symmetries $K_n^2(f)$ and $\tau_n(f)$. Note that the generalized Virasoro type symmetry algebras (30) and (33) are valid not only for $n, m \geqslant 0$, but also for $n, m < 0$ because Φ, the recursion operator of AKNS, is invertible [19] and K_n^2 and τ_n^1 are not equal to zero for $n < 0$. As a comparison, in some other integrable models, like the KP and Toda equations [5, 6], the generalized W_∞ algebra exists only for $n, m \geqslant 0$.

If we take the arbitrary function $f(y)$ as a polynomial of y, say y^n, the generalized Virasoro algebras constituted by $K_n^2(f)$ and $\tau_n(f)$ reduce to

$$
[W_n^r, W_m^s] = (s-r)w_{m+n}^{s+r-1} \qquad (w_n^r \equiv K_n^2(y^r)) \qquad m, n, r, s = 0, \pm 1, \pm 2 \ldots \tag{35}
$$

and

$$
[V_n^r, V_m^s] = (m-n+s-r)V_{m+n-1}^{r+s} \qquad (V_n^r \equiv \tau_n(y^r)) \qquad m, n, r, s = 0, \pm 1, \pm 2, \ldots. \tag{36}
$$

Furthermore, the algebra (35) and (36) will reduce to the usual Virasoro algebra for $m = n = 0$ and $r = s = 0$ respectively. W_n^r and V_n^r are truncated for $r \geqslant 0$. If we take the arbitrary function $f(y)$ as the exponential of y say, $f(y) = \exp ry$ $(r = 0, \pm 1, \pm 2, \ldots)$, the generalized Virasoro algebra constituted by $K_n^2(f)$ reduces to

$$
[W_n^r, W_m^s] = (s-r)W_{m+n}^{s+r} \qquad n, m, r, s = 0, \pm 1 \pm 2, \ldots \tag{37}
$$

where $W_n^r \equiv K_n^2(\exp ry)$ are not truncated for all non-zero r. The algebra (37) will also reduce to the usual Virasoro algebra for $m = n = 0$, whereas the algebra (33) is not closed for $f(y) = \exp ry$. Moreover, if we restrict $f(y)$ as

$$
f(y) = y^r \exp sy \qquad (r, s = 0, \pm 1, \pm 2, \ldots) \tag{38}
$$

then the algebras (30) and (34) reduce to the 'coloured' Virasoro algebras ($W_n^{r_1,s_1} \equiv K_n^2(y^r \exp sy)$, $V_n^{r,s} \equiv \tau_n(y^r \exp sy)$, $n, r, s = 0, \pm 1, \pm 2, \ldots$)

$$[W_{n_1}^{r_1,s_1}, W_{n_2}^{r_2,s_2}] = (r_2 - r_1) W_{n_1+n_2}^{r_1+r_2-1,s_1+s_2} + (s_2 - s_1) W_{n_1+n_2}^{r_1+r_2,s_1+s_2} \tag{39}$$

$$[V_{n_1}^{r_1,s_1}, V_{n_2}^{r_2,s_2}] = (n_2 - n_1 + r_2 - r_1) V_{n_1+n_2-1}^{r_1+r_2,s_1+s_2} + (s_2 - s_1) V_{n_1+n_2-1}^{r_1+r_2+1,s_1+s_2} \tag{40}$$

where all $W_n^{r,s} \equiv K_n^2(y^r \exp sy)$ and $V_n^{r,s} \equiv \tau_n(y^r \exp sy)$ are not truncated for $s \neq 0$.

4. Summary and discussion

The breaking soliton equation is another type of $(2+1)$-dimensional integrable model. Any $(1+1)$-dimensional integrable model which has one hereditary recursion operator can be extended to a $(2+1)$-dimensional breaking soliton equation. The AKNS-type $(2+1)$-dimensional breaking soliton equation is a typical one amongst them. In this paper we have studied the symmetry structure of the AKNS-type breaking soliton equation by means of the formal series symmetry approach. Starting from every one of the known symmetries, one can obtain a new symmetry with an arbitrary function of y. Generally, this new symmetry exhibits as a non-truncated series. If the arbitrary function is fixed as a polynomial of y, then the formal series is truncated naturally. Using the known simple symmetries given by Li [16, 18], we get three sets of generalized infinitely many formal series symmetries. All the spacetime polynomial dependent symmetries can be obtained from these formal series symmetries simply by taking the arbitrary functions of these symmetries as polynomials. From [5–8], we know that the truncated symmetries obtained from the formal series symmetries constitute usually the generalized W_∞ algebra, a type of generalization of the usual Virasoro algebra, whereas the non-truncated symmetries of the breaking soliton equation constitute another type of generalization of the Virasoro algebra. For the AKNS-type breaking soliton equation, two types of generalized Virasoro algebras constituted by two sets of non-truncated symmetries are found. If we fix the arbitrary functions as the polynomials or exponentials of y, then some types of usual Virasoro algebras can be obtained as some special subalgebras.

Though the discussion of this paper is confined to the case of the AKNS-type breaking soliton equation, many of the results remain valid for all breaking soliton equations. For instance, symmetries $K_n^1(f)$ and $K_n^2(f)$ shown by equations (21) and (22) are valid for all breaking soliton equations whence the recursion operators of them are space (x, y) translation invariant and then the generalized symmetry algebras (28)–(30) constituted by $K_n^1(f)$ and $K_n^2(f)$ are also valid for all (x, y)-translation invariant breaking soliton hierarchies. The non-truncated formal series symmetries are worthy of further study especially in other types of integrable model.

Acknowledgments

The author would like to thank the referees for useful comments. The work was supported by the National Nature Science Foundation of China and the Natural Science Foundation of Zhejiang Province in China. The author would also like to thank Professors G-j Ni, Q-p Liu and X-f Chen for their help and encouragement.

References

[1] Park Q H 1990 *Nucl. Phys.* B 33 267

Avan J 1992 *Phys. Lett.* **168A** 363

[2] Yamagishi K 1991 *Phys. Lett.* **259B** 436

[3] Hoppe J 1982 *MIT PhD Thesis*; 1987 *Proc. Int. Workshop on Constraints Theory and Relativistic Dynamics* ed G Longhi and L Lusanna (Singapore: World Scientific)

[4] Bial A and Gervais J L 1989 *Nucl. Phys.* B **326** 222

[5] Lou S-y 1993 *Phys. Rev. Lett.* **71** 4099
Lou S-y and Qian X-m 1994 *J. Phys. A: Math. Gen.* **27** L641

[6] Lou S-y 1993 *J. Phys. A: Math. Gen.* **26** 4387
Lou S-y and Lin J 1994 *Phys. Lett.* **185A** 29

[7] Lou S-y 1994 *J. Phys. A: Math. Gen.* **27** 3235

[8] Lou S-y 1994 *J. Math. Phys.* **35** 1755

[9] Lou S-y and Hu X-b 1994 *J. Phys. A: Math. Gen.* **27** L207

[10] Han P and Lou S-y 1994 *Commun. Theor. Phys.* **22** 437

[11] Lou S-y, Yu J, Wen J-p and Qian X-m 1994 *Acta Phys. Sin.* **43** 1050 (in Chinese)

[12] Bogoyovlenskii O I 1990 *Usp Mat. Nauk.* **45** 17; 1989 *Izv. Akad. Nauk. SSSR Ser. Mat.* **53** 234, 907; 1989 *Izv. Akad. Nauk. SSSR Ser. Mat.* **54** 1123

[13] Strachan I A B 1992 *Inverse Problems* **8** L21

[14] Strachan I A B 1993 *J. Math. Phys.* **34** 243

[15] Li Y-s 1994 *Preprint*

[16] Li Y-s 1993 *Int. J. Mod. Phys. A (Proc. Suppl.)* **3A** 523

[17] Orlov A Yu and Schulman E I 1986 *Lett. Math. Phys.* **12** 171

[18] Li Y-s and Zhang Y-j 1993 *J. Phys. A: Math. Gen.* **26** 7487

[19] Lou S-y and Chen W-z 1993 *Phys. Lett.* **179A** 271